GLOBAL SYNCHRONIZATION OF COUPLED DELAYED NEURAL NETWORKS AND APPLICATIONS TO CHAOTIC CNN MODELS*

GUANRONG CHEN
Department of Electronic Engineering, City University of Hong Kong, P. R. China

JIN ZHOU
Institute of Mathematics, Fudan University, Shanghai 200433, P. R. China
Department of Applied Mathematics, Hebei University of Technology, Tianjin 300130, P. R. China

ZENGRONG LIU
Department of Mathematics, Shanghai University, Shanghai 200436, P. R. China

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This paper formulates the model and then studies its dynamics of a system of linearly and diffusively coupled identical delayed neural networks (DNNs), which is generalization of delayed Hopfied neural networks (DHNNs) and delayed cellular neural networks (DCNNs). In particular, a simple yet generic sufficient condition for global synchronization of such coupled DNNs is derived based on the Lyapunov functional methods and Hermitian matrix theory. It is shown that global synchronization of coupled DNNs is ensured by a suitable design of the coupling matrix and the inner linking matrix. Furthermore, the result is applied to some typical chaotic neural networks. Finally, numerical simulations are presented to demonstrate the effectiveness of the approach.

Keywords: Chaos; synchronization; delayed neural networks; chaotic cellular neural networks.

1. Introduction
Recently, there is increasing interest in the study of dynamical properties of delayed neural networks (DNNs). Most previous studies have predominantly concentrated on the stability analysis and periodic oscillations of this kind of networks [Cao, 2000; Lu, 2001; Zhang, 2002]. However, it has been shown that such networks can exhibit some complicated dynamics and even chaotic behaviors. In particular, the introduction of delays into neural networks makes their dynamical behaviors much more complicated, for example, there are strange attractors even in first-order continuous-time autonomous delayed Hopfied neural networks [Lu, 2002; Lu & He, 1996; Ruan et al., 2002; Zou & Nossek, 1993]. On the other hand, experimental and theoretical studies have revealed that a mammalian brain not only can display in its dynamical behavior strange attractors and other transient characteristics for its associative memories [Hjelmfelt & Ross, 1994; Skarda & Freeman, 1987], but also can modulate oscillatory neuronal synchronization by selective

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visual attention [Fire et al., 2001; Steinmetz et al., 2000]. Therefore, investigation of such dynamical properties is indispensable for practical design and applications of delayed neural networks.

Synchronization of coupled chaotic systems has also been intensively investigated in the last decade due to its potential applications in various fields including chaos generators design, secure communication, chemical reactions, biological systems, information science, etc. There are many important results focusing mainly on some well-known chaotic systems such as the Lorenz system, Rössler system, Chua system, Chen system, and so on (see e.g. [Chen & Dong, 1998; Jiang et al., 2003; Pecora & Carroll, 1997; Zhou et al., 2002] and the references therein).

It has been noticed that many studies of chaos synchronization only provide local convergence analysis via the linearization techniques, i.e. using local Jacobian matrix analysis [Pecora, 1998; Pecora & Carroll, 1998; Rangarajan & Ding, 2002; Wang & Chen, 2002a, 2002b]. However, there are many limitations in applying linearization techniques: they are strictly local and usually can only be applied to autonomous systems, not to mention the fact that many chaotic systems cannot be globally linearized such as Chua’s circuit and chaotic cellular neural networks. For practical reasons, it is always desirable to derive some simple, less conservative, and more efficient criteria for global chaos synchronization particularly regarding engineering applications. In addition, literature dealing with chaos synchronization of coupled systems with delays appears to be scarce due to the difficulties in mathematical analysis for higher-dimensional nonlinear delayed systems. This motivates the present investigation of global synchronization for higher-dimensional coupled delayed neural networks, which precisely addresses this difficult issue.

In this paper, we are interested in synchronization phenomena in a system composing of identical delayed neural networks, which is a generalization of delayed Hopfied neural networks and delayed cellular neural networks, and is coupled in arbitrary array via linear diffusive coupling [Pogromsky & Nijmeijer, 2001; Rangarajan & Ding, 2002; Wang & Chen, 2002a, 2002b; Wu & Chua, 1995]. Based on the Lyapunov functional method and Hermitian matrices theory [Chen, 2000; Hale, 1997; Pogromsky & Nijmeijer, 2001; Wu & Chua, 1995], a simple but less-conservative criterion is derived for global synchronization of such coupled delayed neural networks. It is shown that global synchronization of coupled delayed neural networks is ensured by a suitable design of the coupling matrix and the inner linking matrix. To this end, the theoretical results will be illustrated by computer simulations on coupled chaotic cellular neural networks.

This paper is organized as follows. In Sec. 2, some necessary background materials are presented, and a simple configuration of coupled delayed neural networks is formulated. Section 3 deals with synchronization stability analysis for coupled delayed neural networks. In this section, a simple yet generic criterion is established for global synchronization of such coupled delayed neural networks, based on rigorous mathematical analysis. Section 4 discusses a typical class of chaotic cellular networks, where some simple sufficient conditions on the coupling matrix and the inner linking matrix for achieving global synchronization are obtained. In this section, numerical simulations are worked out to demonstrate the effectiveness of the approach. Finally, some concluding remarks are given in Sec. 5.

2. Preliminaries

In this section, some concepts and results from [Chen, 2000; Hale, 1997; Pogromsky & Nijmeijer, 2001; Wu & Chua, 1995] are given, which will be used throughout the paper.

First, consider the following retarded dynamical system:

$$\frac{dx(t)}{dt} = f(t, x_t),$$

where $x(t) \in R^n$, $f : R \times C \rightarrow R^n$ is a functional defined on $R \times C$, with $C \equiv C([-\tau, 0], R^n)$ being the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $R^n$ with norm $\|\phi\| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|$, in which $|\phi(\theta)|$ is the $l_2$ norm of $R^n$, i.e. $|\phi(\theta)| = (\sum_{1 \leq i \leq n} |\phi_i(\theta)|^2)^{\frac{1}{2}}$, and $x_t \in C$ is defined as $x_t(\theta) = x(t + \theta)$ for any $\theta \in [-\tau, 0]$. For this system of functional differential equations (1), its initial condition is an element of $C$. Under some conditions on $f$ (see [Hale, 1997]) for any given $\phi \in C$ and $t_0 \in R$, there exists a unique solution of (1) for the given initial condition $(t_0, \phi)$, and this solution is denoted as $x(t_0, \phi)$.

Let the solution space of system (1) be $X = \{x(t_0, \phi) | (t_0, \phi) \in R \times C\}$, which is also an Banach space by the uniqueness of solution of (1) with
respect to the initial condition. Let \( \Lambda \) be a nonempty compact subset in \( X \), with \( 0 \in \Lambda \). For this set \( \Lambda \), let
\[
|x|_\Lambda = \inf_{\eta \in \Lambda} \text{dist}(x, \eta)
\]
denote the distance from \( x \in X \) to \( \Lambda \).

Here, a brief definition, similarly to that of [Pogromsky & Nijmeijer, 2001], is given:

**Definition 1.** For the retarded dynamical system (1), an invariant set \( \Lambda \in X \) with respect to (1) is said to be *Lyapunov stable* if there exist two numbers, \( Q \geq 1 \) and \( \alpha > 0 \), such that for any \( (t_0, \phi) \), the solution \( x(t_0, \phi) \) satisfies
\[
|x(t_0, \phi)|_\Lambda \leq Q|\phi|_\Lambda e^{-\alpha(t-t_0)}.
\]

Next, some equivalent nonsingular \( M \)-matrix definitions are listed as follows [Chen, 2000].

**Definition 2.** Let \( A \) be a square matrix with nonpositive off-diagonal elements. Then, each of the following conditions is equivalent to the statement “\( A \) is a nonsingular \( M \)-matrix”:

1. The leading principal minor of \( A \) is positive.
2. There is a vector \( x \) (or \( y \)) whose elements are all positive such that the elements of \( Ax \) (or \( A^T y \)) are all positive.
3. The real parts of all the eigenvalues of \( A \) are positive.
4. There is a diagonal matrix, \( P = \text{diag}(p_1, p_2, \ldots, p_n) \), with \( p_i > 0 \), \( i = 1, 2, \ldots, n \), such that \( PA + A^T P \) is a positive definite matrix.

Now, consider an isolate delayed neural network (DNNs), which is described by the following set of differential equations with delays [Cao, 2000; Lu, 2001; Zhang, 2002]:

\[
\begin{align*}
\dot{x}_r(t) &= -c_r x_r(t) + \sum_{s=1}^{n} a_{rs}^0 f_s(x_s(t)) \\
&\quad + \sum_{s=1}^{n} a_{rs}^s f_s(x_s(t - \tau_{rs})) \\
&\quad + u_r(t), \quad r = 1, \ldots, n,
\end{align*}
\]

or, in a compact form,
\[
\dot{x}(t) = -Cx(t) + Af(x(t)) + A^T f(x(t - \tau)) + u(t),
\]

where \( x(t) = (x_1(t), \ldots, x_n(t))^T \in \mathbb{R}^n \) is the state vector of the neural network, \( C = \text{diag}(c_1, \ldots, c_n) \) is a diagonal matrix with positive diagonal entries, \( c_r > 0, \quad r = 1, \ldots, n \), \( A = (a_{rs}^0)_{n \times n} \) is a weight matrix, \( A^T = (a_{rs}^s)_{n \times n} \) is the delayed weight matrix, \( u(t) = (u_1(t), \ldots, u_n(t))^T \in \mathbb{R}^n \) is the input vector function, \( \tau(r) = (\tau_{rs}) \) with the delays \( \tau_{rs} \geq 0, \quad r, s = 1, 2, \ldots, n \), and \( f(x(t)) = [f_1(x_1(t)), \ldots, f_n(x_n(t))]^T \).

Throughout this paper, assume that each of the activation functions \( f_i(x) \) is globally Lipschitz continuous [Cao, 2000; Lu, 2001; Zhang, 2002], i.e. either \( (A_1) \) or \( (A_2) \) is satisfied:

- **(A_1)** There exist constants \( k_r > 0, \quad r = 1, 2, \ldots, n \), such that
  \[
  0 \leq \frac{f_r(x_1) - f_r(x_2)}{x_1 - x_2} \leq k_r, \quad r = 1, 2, \ldots, n,
  \]
  for any two different \( x_1, x_2 \in \mathbb{R} \).
- **(A_2)** There exist constants \( k_r > 0, \quad r = 1, 2, \ldots, n \), such that
  \[
  |f_r(x_1) - f_r(x_2)| \leq k_r|x_1 - x_2|, \quad r = 1, 2, \ldots, n,
  \]
  for any two different \( x_1, x_2 \in \mathbb{R} \).

Obviously, Condition \( (A_2) \) is less conservative than \( (A_1) \), and some general activation functions in conventional neural networks, such as the standard sigmoidal functions and piecewise-linear functions, satisfy Condition \( (A_1) \) or \( (A_2) \).

Finally, a configuration of the coupled delayed neural networks (DNNs) is formulated. Consider a dynamical system consisting of \( N \) linearly and diffusively coupled identical delayed neural networks. The state equations of this system are
\[
\dot{x}_i(t) = -Cx_i(t) + Af(x_i(t)) + A^T f(x_i(t - \tau))
+ u(t) + \sum_{j=1}^{N} b_{ij} x_j(t), \quad i = 1, 2, \ldots, N,
\]
where \( x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{im}(t))^T \in \mathbb{R}^n \) are the state variables of the \( i \)th delayed neural network, \( \Gamma = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{R}^{n \times n} \) is a diagonal matrix, which means that if \( \gamma_i \neq 0 \) then two coupled networks are linked through their \( i \)th state variables, and the coupling matrix \( B = (b_{ij}) \in \mathbb{R}^{N \times N} \) represents the coupling configuration of the system. For simplicity, assume that the coupling is symmetric, i.e. \( b_{ij} = b_{ji} \) and \( b_{ij} \geq 0 \) \( (i \neq j) \) but not all zero, with the diffusive coupling connection [Pogromsky & Nijmeijer, 2001; Rangarajan & Ding, 2002; Wang & Chen, 2002a, 2002b; Wu & Chua, 1995]:
\[
\sum_{j=1}^{N} b_{ij} = \sum_{j=1}^{N} b_{ji} = 0, \quad i = 1, 2, \ldots, N.
\]
This implies that the coupling matrix $B$ is a symmetric irreducible matrix. In this case, it can be shown that zero is an eigenvalue of $B$ with multiplicity 1 and all the other eigenvalues of $B$ are strictly negative.

3. Synchronization Stability Analysis

The dynamical network (4) is said to achieve (asymptotical) synchronization if

$$x_1(t) \to x_2(t) \to \cdots x_n(t) \to s(t), \quad \text{as } t \to \infty$$

(6)

Clearly, the diffusive coupling condition (5) ensures the synchronization state be a solution $s(t) \in \mathbb{R}^n$ of an individual delayed neural network (3)' (asymptotical) synchronization (synchronization manifold):

$$
\Lambda = \{x_1, x_2, \ldots, x_N \in \mathbb{R}^n | x_i = x_j; i, j = 1, 2, \ldots, N. \} 
$$

(8)

If this is the case for all initial conditions in some open neighborhood of $\Lambda$, then synchronization is equivalent to the attractivity of $\Lambda$. The case when $\Lambda$ attracts solutions from a set of positive Lebesgue measure but $\Lambda$ is not Lyapunov stable is called weak synchronization [Pogromsky & Nijmeijer, 2001]. For practical reasons, it is convenient to consider a stronger version where $\Lambda$ is not only attractive but also Lyapunov stable. This case is called strong synchronization [Pogromsky & Nijmeijer, 2001]. In the sequel, only strong synchronization is studied, for the system (4) of diffusively coupled identical networks.

Based on the Lyapunov functional method and Hermitian matrices theory, the following sufficient condition for strong synchronization of the coupled network (4) is first established.

\textbf{Theorem 1.} Consider the dynamical system of diffusively coupled identical network (4). Let the eigenvalues of its coupling matrix $B$ be ordered as

$$0 = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N.$$

(9)

If one of the following conditions is satisfied, then the dynamical system (4) will achieve strong synchronization, i.e. there is an invariant diagonal hyperplane $\Lambda$ (synchronization manifold), which is not only attractive but is also Lyapunov stable.

(1) Assume that $(A_1)$ holds, and there exist $n$ positive numbers $p_1, \ldots, p_n$ and two positive numbers $r_1 \in [0, 1], r_2 \in [0, 1]$, with

$$\lambda(\gamma_i) = \begin{cases} 
\lambda_2, & \text{if } \gamma_i > 0, \\
0, & \text{if } \gamma_i = 0, \\
\lambda_N, & \text{if } \gamma_i < 0,
\end{cases}$$

(10)

and

$$\alpha_i \overset{\text{def}}{=} -c_i p_i + \sum_{j=1}^{n} (p_j |a_{ij}^0| k_{ij}^{2r_1} + p_j |a_{ij}^0| k_{ij}^{2r_2}) + \frac{1}{2} \sum_{j=1}^{n} (p_j |a_{ij}^0| k_{ij}^{2r_2} + p_j |a_{ij}^0| k_{ij}^{2r_2}),$$

such that for all $i = 1, 2, \ldots, n$,

$$|a_{ii}^0| p_i k_i + \alpha_i + p_i \gamma_i \lambda(\gamma_i) > 0,$$

(11)

where $(a_{ii}^0)^+ = \max\{a_{ii}^0, 0\}$.

(2) Assume that $(A_2)$ holds, and for all $i = 1, 2, \ldots, n$,

$$|a_{ii}^0| p_i k_i + \alpha_i + p_i \gamma_i \lambda(\gamma_i) < 0.$$

(12)

Clearly, Condition (11) is less conservative than (12) since $(a_{ii}^0)^+ \leq |a_{ii}^0|$ in general.

\textit{Proof of Theorem 1.} (1) Clearly, by Condition $(A_1)$, the existence and uniqueness of a solution of (4) follows from the Lipschitz continuity of the right-hand side of (4).

Since $B$ is a symmetric and irreducible matrix, by Lemma 6 of [Wu & Chua, 1995], there exists an $L \times N$ matrix $M = (m_{ij})_{L \times N} \in M_2^N(1)$, such that

$$B = -M^T M,$$

where $M_2^N(1)$ is a set of matrices, which is composed of $L \times N$ matrices, defined follow. Each row (for instance, the $i$th row) of $M \in M_2^N(1)$ has exactly one entry $\beta_i$ and one $-\beta_i$, where $\beta_i \neq 0$. All other entries are zeroes. Moreover, for any pair of indices $i$ and $j$, there exist indices $j_1, \ldots, j_l$, where
According to the structure of \( \mathbf{M} \) and \( y(t) = \mathbf{M}x(t) \), one can use \(|y| = |\mathbf{M}x|\) to measure the distance from \( x \in X \) to the invariant hyperplane \( \Lambda \) (synchronization manifold), and the following equalities are easy to verify:

\[
\mathbf{M}u(t) = 0, \quad \mathbf{MC} = C_1 \mathbf{M}, \quad \mathbf{MA} = A_1 \mathbf{M}, \quad \mathbf{MA}^\tau = A_1^\tau \mathbf{M}
\]

and

\[
\mathbf{M}f(x) \overset{\text{def}}{=} g(y)
\]

\[
= (g(y_1), g(y_2), \ldots, g(y_L))^\top
\]

\[
= \left( \beta_1 \left( f(x_{11}) - f \left( x_{11} - \frac{1}{\beta_1} y_1 \right) \right), \ldots, \beta_L \left( f(x_{L1}) - f \left( x_{L1} - \frac{1}{\beta_L} y_L \right) \right) \right)^\top,
\]

where \( L_i \in \{1, 2, \ldots, N\}, i = 1, 2, \ldots, L \).

From Condition (A1), \( g_i(i = 1, 2, \ldots, L) \) possesses the following properties:

\[
|g_i(y_{ir})| \leq k_r |y_{ir}|, \quad 0 \leq y_{ir} g_i(y_{ir}) \leq k_r y_{ir}^2,
\]

for any \( y_{ir}, i = 1, 2, \ldots, L, r = 1, 2, \ldots, n \).

It follows from Condition (11) that there exists a small real number \( \varepsilon > 0 \), with

\[
\alpha'_r \overset{\text{def}}{=} \left( -c_r + \frac{\varepsilon}{2} \right) p_r + \frac{1}{2} \sum_{s=1}^{n} (p_r |a_{rs}^0| k_s^{2r_1} + p_s |a_{sr}^0| k_r^{2(1-r_1)})
\]

\[
+ \frac{1}{2} \sum_{s=1}^{n} (p_r |a_{rs}^\tau| k_s^{2r_2} + p_s |a_{sr}^\tau| k_r^{2(1-r_2)} e^{\varepsilon \gamma_s}), \quad r = 1, 2, \ldots, n,
\]

such that

\[
(a_{rs}^0)^+ p_r k_r + \alpha'_r + p_r \gamma_r \lambda(\gamma_r) < 0, \quad r = 1, 2, \ldots, n.
\]

Let \( P = \text{diag}(p_1, p_2, \ldots, p_n) \), \( \mathbf{P} = I_L \otimes P \). Then one can construct a Lyapunov functional of the following form with respect to system (13):

\[
V(x)(t) = \frac{1}{2} x^\top(t) \mathbf{M}^\top \mathbf{P} x(t) e^{\varepsilon t} + \frac{1}{2} \sum_{i=1}^{L} \sum_{r=1}^{n} p_r |a_{rs}^\tau| k_s^{2(1-r_2)} \int_{t-r_s}^{t} y_{ir}^2(u) e^{\varepsilon (u + \gamma_s) \lambda(\gamma_s)} du.
\]

From the construction of the above Lyapunov functional, it is easy to see that

\[
V = V(x)(t) = \tilde{V}(y)(t) = \sum_{i=1}^{L} V_i(y_i)(t),
\]
where

\[
V_i(y_i(t)) = \frac{1}{2} y_i^T(t) P y_i(t) e^{\varepsilon t} + \frac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} p_r |a_{rs}^T| k_s^{2(1-\tau_r)} \int_{t - \tau_r}^{t} y_{is}(u)e^{\varepsilon(u+\tau_r)} \, du
\]

\[
= \sum_{r=1}^{n} p_r \left\{ \frac{1}{2} y_{ir}(t)e^{\varepsilon t} + \frac{1}{2} \sum_{s=1}^{n} |a_{rs}^T| k_s^{2(1-\tau_r)} \int_{t - \tau_r}^{t} y_{is}(u)e^{\varepsilon(u+\tau_r)} \, du \right\},
\]

or, with the initial condition,

\[
V_i(t, \phi_i) = \sum_{r=1}^{n} p_r \left\{ \frac{1}{2} \phi_{ir}^2(0)e^{\varepsilon t} + \frac{1}{2} \sum_{s=1}^{n} |a_{rs}^T| k_s^{2(1-\tau_r)} \int_{t - \tau_r}^{t} \phi_{is}^2(u-t)e^{\varepsilon(u+\tau_r)} \, du \right\}.
\]

Differentiating \( V \) with respect to time along the solution of (13), by some elementary computations and the conditions (14), (15) and (16), one has

\[
\frac{dV}{dt} |_{(13)} = \frac{1}{2} x^T(t) M^T P M x(t) e^{\varepsilon t} + x^T(t) M^T P M [-C x(t) + A f(x(t)) + A^T f(x(t - \Gamma))] + u(t) + B x(t) e^{\varepsilon t}
\]

\[
+ \frac{d}{dt} \left[ \frac{1}{2} \sum_{i=1}^{L} \sum_{r=1}^{n} \sum_{s=1}^{n} p_r |a_{rs}^T| k_s^{2(1-\tau_r)} \int_{t - \tau_r}^{t} y_{is}^2(u)e^{\varepsilon(u+\tau_r)} \, du \right]
\]

\[
= \left\{ \frac{1}{2} x^T(t) P y(t) e^{\varepsilon t} + y^T(t) P [-C_1 y(t) + A_1 M f(x(t)) + A_1 M^T f(x(t - \Gamma))] e^{\varepsilon t}
\]

\[
+ \frac{1}{2} \frac{d}{dt} \left[ \sum_{i=1}^{L} \sum_{r=1}^{n} \sum_{s=1}^{n} p_r |a_{rs}^T| k_s^{2(1-\tau_r)} \int_{t - \tau_r}^{t} y_{is}^2(u)e^{\varepsilon(u+\tau_r)} \, du \right] \right\} + x^T(t) M^T P M x(t) e^{\varepsilon t}
\]

\[
\leq \sum_{i=1}^{L} \sum_{r=1}^{n} p_r \left\{ -c_r y_{ir}^2(t) e^{\varepsilon t} + \frac{\varepsilon}{2} y_{ir}^2(t) e^{\varepsilon t} + \sum_{s=1}^{n} a_{rs}^0 g_s(y_{is}(t)) + \sum_{s=1}^{n} a_{rs}^e g_s(y_{is}(t - \tau_r)) \right\} e^{\varepsilon t}
\]

\[
+ \frac{1}{2} \sum_{s=1}^{n} |a_{rs}^T| k_s^{2(1-\tau_r)} (y_{is}^2(t)e^{\varepsilon \tau_r} - y_{is}^2(t - \tau_r)e^{\varepsilon t}) \right\} + x^T(t) M^T P M x(t) e^{\varepsilon t}
\]

\[
\leq \sum_{i=1}^{L} \sum_{r=1}^{n} p_r \left\{ -c_r + \frac{\varepsilon}{2} y_{ir}^2(t) e^{\varepsilon t} + a_{rs}^0 g_r(y_{ir}(t)) e^{\varepsilon t} + \sum_{s=1}^{n} |a_{rs}^0||y_{ir}(t)|k_s|y_{is}(t)||e^{\varepsilon t}
\]

\[
+ \sum_{s=1}^{n} |a_{rs}^T||y_{is}(t)|k_s|y_{is}(t - \tau_r)||e^{\varepsilon t} + \frac{1}{2} \sum_{s=1}^{n} |a_{rs}^T| k_s^{2(1-\tau_r)} (y_{is}^2(t)e^{\varepsilon \tau_r} - y_{is}^2(t - \tau_r)e^{\varepsilon t}) \right\} + x^T(t) M^T P M x(t) e^{\varepsilon t}
\]
\[
\begin{align*}
&\leq \sum_{i=1}^{L} \sum_{r=1}^{n} \left\{ a_{rr}^0 p_r y_{ir}(t) g_r(y_{ir}(t)) e^{\varepsilon t} + \left[ -c_r + \frac{\varepsilon}{2} \right] p_r + \frac{1}{2} \sum_{s=1}^{n} (p_r |a_{rs}^0|^2 k_s^2 + p_s |a_{sr}^0|^2 k_r^{2(1-r)_{mr}}) \right\} \\
&\quad + \frac{1}{2} \sum_{s=1}^{n} (p_r |a_{rs}^0|^2 k_s^2 + p_s |a_{sr}^0|^2 k_r^{2(1-r)_{mr}} e^{\varepsilon t_{rs}}) \right\} + x^\top(t) M^T P B x(t) e^{\varepsilon t} \\
&\leq \sum_{i=1}^{L} \sum_{r=1}^{n} \left\{ (a_{rr}^0 - p_k + \alpha_i') y_{ir}^2(t) e^{\varepsilon t} + x^\top(t) M^T P B x(t) e^{\varepsilon t} \right\} \\
&= \sum_{i=1}^{n} \left\{ ((a_{ii}^0)^2 - p_k k_i + \alpha_i') x_i^2(t) e^{\varepsilon t} + x^\top(t) M^T P B x(t) e^{\varepsilon t} \right\}
\end{align*}
\]

Notice that \( \overline{P}_j(t) = M \overline{P}_j(t), j = 1, 2, \ldots, n, \)

Only if the largest nonzero eigenvalues of the symmetric matrices \( p_i \gamma_i B \) are less than \(-((a_{ii}^0)^2 - p_k k_i + \alpha_i'). \)

Since the largest nonzero eigenvalues of \( p_i \gamma_i B \) are \( p_i \lambda(\gamma_i), \) it follows from (11) that

\[
\frac{dV}{dt} \bigg|_{(13)} \leq 0.
\]

This implies that

\[
V(t) \leq V(0), \quad \text{for all } t \geq 0.
\]

The above construction of Lyapunov functional implies that

\[
\frac{1}{2} \sum_{i=1}^{L} \sum_{r=1}^{n} \left\{ p_r \right\} e^{\varepsilon t} \sum_{i=1}^{n} \left\{ y_{ir}^2(t) \right\}
\]

and from (18), (19) and (20), one has

\[
V(0) = \sum_{i=1}^{L} \sum_{r=1}^{n} p_r \left\{ \frac{1}{2} \phi_{ir}^2(0) + \frac{2}{2} \sum_{s=1}^{n} |a_{rs}^0|^2 k_s^2(1-r_{sr}) \int_{-\tau_{rs}}^{0} \phi_{ir}^2(u) e^{\varepsilon(u+\tau_{rs})} du \right\}
\]

\[
\leq \frac{1}{2} \max_{1 \leq r \leq n} \left\{ p_r \right\} \sum_{i=1}^{L} \sum_{r=1}^{n} \phi_{ir}^2(0) + \frac{1}{2} \sum_{i=1}^{L} \sum_{r=1}^{n} \sum_{s=1}^{n} p_r |a_{rs}^0|^2 k_s^2(1-r_{sr}) e^{\varepsilon \tau_{rs}} \int_{-\tau_{rs}}^{0} \phi_{ir}^2(u) du
\]

\[
\leq \frac{1}{2} \max_{1 \leq r \leq n} \left\{ p_r \right\} \sum_{i=1}^{L} \left\| \phi_i \right\|^2 + \max_{1 \leq s \leq n} \left\{ \frac{1}{2} \sum_{r=1}^{n} p_r |a_{rs}^0|^2 k_s^2(1-r_{sr}) e^{\varepsilon \tau_{rs}} \right\} \sum_{i=1}^{L} \int_{-\tau_{rs}}^{0} \sum_{s=1}^{n} \phi_{ir}^2(u) du
\]

\[
= \left( \frac{1}{2} \max_{1 \leq r \leq n} \left\{ p_r \right\} + \max_{1 \leq s \leq n} \left\{ \frac{1}{2} \sum_{r=1}^{n} p_r |a_{rs}^0|^2 k_s^2(1-r_{sr}) e^{\varepsilon \tau_{rs}} \right\} \right) \sum_{i=1}^{L} \left\| \phi_i \right\|^2,
\]
Combining inequalities (24), (25) and (26), one has
\[ |y(t)| = \left( \sum_{i=1}^{L} y_i^2(t) \right)^{\frac{1}{2}} = \left( \sum_{i=1}^{L} \sum_{r=1}^{n} y_{ir}^2(t) \right)^{\frac{1}{2}} \leq Q \left( \sum_{i=1}^{L} \|\phi_i\|^2 \right)^{\frac{1}{2}} e^{-\frac{\gamma t}{2}} \leq Q\|\phi\|e^{-\frac{\gamma t}{2}}, \] (27)

where
\[ Q = \max \left\{ 1, \left[ \left( \min_{1 \leq r \leq n} \{p_r\} \right)^{-1} \max_{1 \leq r \leq n} \{p_r\} \right] + \max_{1 \leq s \leq n} \left\{ \sum_{r=1}^{n} p_r|a_{rs}^\tau|k_s^{2(1-r_2)}e^{\varepsilon r_s \tau} \right\} \right\}^{\frac{1}{2}}. \]

(2) One can use the same arguments as that in the proof of (1). In fact, from Condition (A2), it is easy to derive (21) if one substitutes \((a_{ii}^0)^+\) by \(|a_{ii}^0|\). The rest of the proof is similar to that of (1), and it is omitted.

In view of (1) and (2), and Definition 1, this completes the proof of Theorem 1. □

For all \(i = 1, 2, \ldots, n\), denote the matrix \(D^1 = (d^1_{ij})_{n \times n}\) with
\[ d^1_{ij} = \begin{cases} (c_i + \gamma \lambda(\gamma_i)) - ((a_{ii}^0)^+ + |a_{ii}^\tau|)k_i, & i = j, \\ -((a_{ij}^0)^+ + |a_{ij}^\tau|)k_j, & i \neq j, \end{cases} \]
and the matrix \(D^2 = (d^2_{ij})_{n \times n}\) with
\[ d^2_{ij} = \begin{cases} (c_i + \gamma \lambda(\gamma_i)) - (a_{ii}^0 + |a_{ii}^\tau|)k_i, & i = j, \\ -(|a_{ij}^0| + |a_{ij}^\tau|)k_j, & i \neq j. \end{cases} \]

It is easy to show that Condition (11) of Theorem 1 is equivalent to
\[ a_{ii}^0p_i + \frac{1}{2} \sum_{j=1}^{n} (p_i|a_{ij}^0|k_j^{2r_1} + p_j|a_{ji}^0|k_i^{2(1-r_1)}) + \frac{1}{2} \sum_{j=1}^{n} (p_i|a_{ij}^\tau|k_j^{2r_2} + p_j|a_{ji}^\tau|k_i^{2(1-r_2)}) \leq (c_i - \gamma \lambda(\gamma_i))p_i, \quad i = 1, 2, \ldots, n. \] (28)

When taking \(r_1 = r_2 = 1/2\), this condition becomes
\[ \frac{1}{2} \sum_{j=1}^{n} (p_i|a_{ij}^0|k_j + p_j|a_{ji}^0|k_i) + \frac{1}{2} \sum_{j=1}^{n} (p_i|a_{ij}^\tau|k_j + p_j|a_{ji}^\tau|k_i) \leq ((c_i - \gamma \lambda(\gamma_i)) - (a_{ii}^0 + |a_{ii}^\tau|)k_i)p_i, \quad i = 1, 2, \ldots, n, \] (29)
which implies that the symmetric matrix \(\frac{1}{2}(PD^1 + (D^1)^T P)\) is positive definite [Chen, 2000]. By Condition (4) of Definition 2, this condition means that \(D^1\) is a nonsingular \(M\)-matrix. Thus, one can obtain a simple but generic condition as follows:

**Corollary 1.** If one of the following conditions is satisfied, then the dynamical system (4) achieves strong synchronization:

1. Condition (A1) holds, and the matrix \(D^1\) is a nonsingular \(M\)-matrix.
2. Condition (A2) holds, and the matrix \(D^2\) is a nonsingular \(M\)-matrix.

From Theorem 1 and Corollary 1, it can be concluded that global synchronization of coupled delayed neural networks (4) can be achieved by a suitable design of the coupling matrix and the inner linking matrix such that the conditions of Theorem 1 or Corollary 1 are satisfied.

### 4. Applications to Chaotic CNN Models

In this section, the above analytic results are illustrated by a chaotic cellular neural network.

Consider a three-dimensional CNN model of the form (3), with
\[ A = \begin{bmatrix} 1.2500 & -3.200 & -3.200 \\ -3.200 & 1.1000 & -4.4000 \\ -3.200 & 4.4000 & 1.000 \end{bmatrix}, \quad A^\tau = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]
and \(u(t) = (0, 0, 0)^T\). The state equations of the network are
\[ \dot{x}(t) = -Cx(t) + Af(x(t)), \] (30)
where $x(t) = (x_1(t), x_2(t), x_3(t))^\top$ is the state vector of the network, and $f(x(t)) = [f(x_1(t)), \ldots, f(x_n(t))]^\top$. Take the activation functions $f_i(x) = f(x)$ ($i = 1, 2, 3$) as the following piecewise-linear functions:

$$f(x) = \frac{1}{2}(|x + 1| - |x - 1|).$$

Clearly, the network is a typical cellular neural network.

Model (30) has three equilibria [Chen & Dong, 1998; Zou & Nosek, 1993]:

$$\mathbf{\tilde{\pi}}_+ = (1.1971, 0.7273, -0.7107) \in \Sigma_+$$

$$= \{(x_1, x_2, x_3)|x_1 \geq 1, |x_2| < 1, |x_3| < 1\},$$

$$\mathbf{\tilde{\pi}}_0 = (0, 0, 0) \in \Sigma_0$$

$$= \{(x_1, x_2, x_3)||x_1| < 1, |x_2| < 1, |x_3| < 1\},$$

$$\mathbf{\tilde{\pi}}_- = (-1.1971, -0.7273, 0.7107) \in \Sigma_-$$

$$= \{(x_1, x_2, x_3)|x_1 \leq -1, |x_2| < 1, |x_3| < 1\}.$$

The model has a chaotic attractor as shown in Fig. 1, which is plotted with the initial condition $(x_1(0), x_2(0), x_3(0)) = (0.1, 0.1, 0.1)$.

Now, consider a dynamical system consisting of three linearly and diffusively coupled identical model (30). The state equations of the entire system are

$$\dot{x}_i(t) = -Cx_i(t) + Af(x_i(t)) + \sum_{j=1}^{3} b_{ij} \Gamma x_j(t), \quad i = 1, 2, 3, \quad (31)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^\top$ are the state variables of the $i$th cellular neural network.
Let the coupling matrix be \( B = \begin{bmatrix} -8 & 2 & 6 \\ 2 & -4 & 2 \\ 6 & 2 & -8 \end{bmatrix} \).

This matrix has eigenvalues 0, -6 and -14. The inner linking matrix is taken as follows:

(i) If one chooses the inner linking matrix \( \Gamma = \rho_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \), then

\[
D^1 = \begin{bmatrix} 0.2500 + 6\rho_1 & -3.2000 & -3.2000 \\ -3.2000 & 0.1000 + 6\rho_1 & -4.4000 \\ -3.2000 & -4.4000 & 6\rho_1 \end{bmatrix}.
\]

It is easy to verify that all the eigenvalues of \( D^1 \) are positive if \( \rho_1 \geq 1.1878 \). This means that \( D^1 \) is a nonsingular \( M \)-matrix, by Condition (3) of Definition 2. According to Corollary 1, \( \rho_1 > 1.1878 \) ensures global synchronization of the coupled network (31) when the network is linked through all state variables. This can be observed from Fig. 2. In the simulation, \( \rho_1 = 1.1878 \) with initial conditions \((x_{11}(0), x_{12}(0), x_{13}(0))^\top = (3.0, 0.2, -0.2)^\top, (x_{21}(0), x_{22}(0), x_{23}(0))^\top = (1.2, -0.2, 2.0)^\top, \) and \((x_{31}(0), x_{32}(0), x_{33}(0))^\top = (-0.2, 2.0, -1.2)^\top. \)

(ii) If one chooses the inner linking matrix \( \Gamma = \rho_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \), then

\[
D^1 = \begin{bmatrix} 0.2500 + 6\rho_2 & -3.2000 & -3.2000 \\ -3.2000 & 0.1000 & -4.4000 \\ -3.2000 & -4.4000 & 6\rho_2 \end{bmatrix}.
\]

It is not difficult to verify that \( D^1 \) is a nonsingular \( M \)-matrix if \( \rho_2 \geq 49.8900 \). Also, it follows
from Corollary 1 that \( \rho_2 \geq 49.8900 \) guarantees global synchronization of the coupled networks (31) when the network is linked through their first and third state variables. The simulation results are shown in Fig. 3, where \( \rho_2 = 49.8900 \) with initial conditions \((x_{11}(0), x_{12}(0), x_{13}(0))^{\top} = (3.0, 0.2, -0.2)^{\top}\), \((x_{21}(0), x_{22}(0), x_{23}(0))^{\top} = (1.2, -0.2, 2.0)^{\top}\), and \((x_{31}(0), x_{32}(0), x_{33}(0))^{\top} = (-0.2, 2.0, -1.2)^{\top}\).

5. Conclusions

In this paper, a general model of a system of linearly and diffusively coupled identical delayed neural networks has been formulated and its dynamics have been studied. In particular, a simple sufficient condition for global synchronization of such coupled delayed neural networks has been derived based on the Lyapunov functional method and Hermitian matrices theory. It is shown that global synchronization of coupled delayed neural networks can be achieved by a suitable design of the coupling matrix and the inner linking matrix. Furthermore, the results can be applied to some typical chaotic neural networks, delayed Hopfied neural networks and delayed cellular neural networks. The theoretical results are illustrated by computer simulations on a typical chaotic cellular neural network, verifying that the obtained conditions are convenient to use for design and applications.

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References
